

Fuzzy Phase Space Structure as Approach to Quantization

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Abstract

Modification of a nonrelativistic phase space structure based on fuzzy ordered sets (Fosets) structure investigated as a possible nonrelativistic quantization framework. In this model particle's m state corresponds to Foset element - fuzzy point. Due to fuzzy ordering its space coordinate x acquires principal uncertainty σ_x . It's shown that proposed Mechanics on fuzzy phase space manifold reproduces the main quantum effects, in particular the interference of quantum states.

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1 Introduction

Quantum Mechanics (QM) is now a well established physical theory, yet its relation with physical space-time structure and relativity isn't quite clear and actively discussed now. This interest enforced by the recent indications that space-time properties at small (Planck) scale can be quite exotic [1, 2]. Due to the absence of any experimental information it seems instructive to look for some indications reconsidering under this angle standard Quantum Physics space-time picture.

The additional attention to QM foundations induced also by Gravity quantization problems. In particular Isham proposed that some space-time properties like the metrics or topological structure, assumed in the standard quantization should be modified significantly or even rejected completely [5]. Our work motivated largely by this ideas in which framework we explore Set structure of space-time manifold M_{s-t} . Remind that for

Euclidean Geometry in 1-dimension its basic elements are points x_i which are ordered, as follows from Euclidean metrics definitions ; i.e. proposition :

$$LP_x : \quad x_1 \leq x_2.or.x_2 \leq x_1$$

is true for arbitrary x_1, x_2 and thus x_i set X is the ordered set. Classical particle m state corresponds to M_{s-t} point $x(t)$ and formally this state is the point $r_p(t)$ in phase space. In QM one regards as particle's state the extended object - Dirac state vector Ψ evolving on the same M_{s-t} manifold ,i.e classical space-time transferred to QM copiously.

Yet Set Theory permit other set structures, for which elements are weakly or fuzzy ordered relative to each other. In our approach to quantization such fuzzy ordered set (Foset) constitutes the basic space-time manifold M_F with fuzzy relations between its elements - fuzzy points for which proposition L_x can be untrue and only a weak propositions of the kind :' x_1 is in x_2 vicinity of the approximate width σ_x ' characterizes a points relations. On this manifold the particles evolves which are 'pointlike' in a sense that their states corresponds to this fuzzy points. If the space coordinate axe X related to some point O can be defined on M_F , then fuzzy points will be smeared on x axe with arbitrary dispersion σ_x . Furthermore the particles mechanics on fuzzy manifold described here permit to reproduce the main quantum effects.

Some years ago it was shown that fuzzy sets formalism can have important QM applications, in particular a fuzzy observables are the natural generalizations of QM observables [9]. Such formalism of 'fuzzy lumps' also was applied in quantum gravity and cosmology studies [11]. In the last years it was shown that some of fuzzy sets formalisms are appropriate also for Quantum Logics ([10] and ref. therein). Yet it exploits nonstandard logics of propositions with multiple outcomes , besides standard yes/no.

Quantum phase space geometry to our knowledge never been investigated in fuzzy sets framework and we try here to make first steps in this direction. Recently the progress of noncommutative geometry and its applications [6] enforced interest to space-time geometric structure [4, 2]. It supposes that space-time coordinates x^i are noncommutative which can be revealed at Planck scale, which as will be shown intersects with fuzzy theory results. Note that the term 'Fuzzy Geometry' sometimes used for Nocommutative Geometry ideas, but here we'll use it in its proper meaning for Foset based manifold. This Fuzzy Geometry approach to QM can be interpreted as the novel quantization formalism i.e. formal transition from Classical mechanics to QM which was formulated already via path integrals, Von Neuman algebra, deformations and other approaches. In our case the quantization regarded as transfer from Classical ordered phase space to fuzzy one.

We don't present here the complete theory of systems evolutions on fuzzy manifolds, rather this text is semiquantitative discussion of the main ideas. Any quantization formalism inevitably intersects with QM interpretation problems, discussed here in the final chapter. For our study the most interesting seems the physical meaning of a quantum state vector and in particular the role of complex numbers in QM.

Any novel QM formalism must include the description of its tests in corresponding measurements. The transition from quantum states to experimental probabilities $P(x)$ is the subject of quantum measurement theory, which up to now doesn't have acknowledged

formulation [13]. Thus one should use some variant of quantum measurement theory to make formalism fully consistent. But as will shown below our model is indifferent to their choice, but most preferable are models which literally use Schrodinger dynamics like MWI and decoherence models [13]. Of them the most preferable seems dual selfdescription formalism [12]. In particular it's important to understand in principle how transition from w^Q to random events (state collapse) occurs. In our paper measurement aspects aren't regarded, restricting only to statistical quantum ensembles for which it's enough to use W^Q . We also use some measurement theory methods alike comparison of mixed and pure states distributions.

2 Fuzzy sets and relations

Let's remind the main properties of fuzzy sets and fuzzy relations. Beside standard 'crisp' sets in Abstract Algebra and Set Theory fuzzy sets A^F also regarded for which verity of propositions alike $a_i \in A^F$ characterized by positive weight $w_i \geq 0$, in place of 1,0 in standard logics [7]. Here a_i are elements of standard set A . We'll regard in our model fuzzy relations R and mappings on standard sets, and thus don't need to describe fuzzy sets properties in details, which can be found elsewhere [8].

Note that any geometry based on topology and set theory premises which stipulates its manifold properties. For example 1- dimensional Euclidean geometry set of elements (points) is the ordered set X of continuum power. Thus relation of the kind $x_i \leq x_j$ (or vice versa) always can be introduced for any pair of its elements in 1-dimensional case, for 3 dimensions they are analogous. Its topology is compatible with differentiable manifold. If metrics also defined then the distance r_{ij} between x_i, x_j can be introduced.

To introduce Fosets and the fuzzy relations R_f remind that in partial ordered set (Poset) P_A some of its elements (but not all in general) ordered by the relation $a_i \leq a_j$, which obeys to standard rules [8]. For example the element $a_k \in P_A$ ordered relative to some a_j but isn't ordered relative to some others a_i - i.e. unrelated denoted as $a_k \wr a_i$. Consider discrete Poset P_A which includes 'test' subset A^0 and 'reference' subset A^1 . In A^1 all elements are ordered and elements indexes grows correspondingly to it so that $a_i \leq a_{i+1}$. A^0 element a'_0 unrelated to some $a_i \in A^1$ and let's suppose that all this a_i belongs to interval $[a_l, a_{l+m}]$, i.e. $a_l \leq a_i, a_i \leq a_{l+m}$. a'_0 ordered to other $a_j \in A^1$ and $a_l \leq a'_0, a'_0 \leq a_{l+m}$, so that a'_0 also belongs to interval a_l, a_{l+m} , but is 'smeared' inside it corresponding to weak P_A structure. Described situation can be interpreted as a_0, a_i are approximately equal up to some arbitrary uncertainty σ_a . To introduce the quantitative measure for it one put in correspondence to each a_i the weight $w_i^0 \geq 0$ with norm $\sum w_i^0 = 1$. Poset P_A on which the fuzzy measure w_i^l defined is Foset [7]. As the result w_i^0 gives more detailed description of fuzzy relations between P_A elements; for example if $w_j^0 \approx 1$ and all others $w_i^0 \approx 0$ it means that a'_0 nearly coincides with a_j . Standard ordering corresponds to $w_j^i = \delta_{ij}$ for equal a_i, a_j and $w_{i+1}^i = w_{i-1}^i = \frac{1}{2}$ for $a_i \in A^1$. If A^0 consists of more than one elements analogous fuzzy relations w_i^j relative to A^1 elements can be defined for all of them. A relations between A^0 elements should be defined additionally; they can be ordered or unrelated independently of their relations to A^1 elements. Thus the Fosets

structure is more strictly ordered than Poset one but weaker than for an ordered set. Thus the FOSet ordering structure is more strict than Poset one but weaker than the standard ordering.

Reference subset A^1 can be substituted by continuous metricized subset (axe) X , and if A^0 properties conserved than for $a'_0 \in [x_l, x_m]$ interval (which can be also infinite) the fuzzy a'_0, x_i relations described by continuous distribution $w^0(x) \geq 0$ with norm $\int w^0 dx = 1$.

If fuzzy point a'_0 confined inside interval $E_x = [x_a, x_b]$ where $w(x) > 0$ then a'_0 fuzziness expressed by the proposition L_0 which will be important for Fuzzy Mechanics :

$$L_0 : \quad \forall \Delta x_i \in E_x; \quad a'_0 \in E_x \cap a'_0 \notin \Delta x_i$$

where $\Delta x_i \in E_x$ is any interval contained inside E_x . Regarded example is most simple and more complicated fuzzy geometric relations exists; for our topics most interesting is the situation when $a'_0 \in E_x$ which consists of the two or more noncrossing intervals $Dx_n = [x_l^n, x_m^n]$ and in this case L_0 is true also. Generalization of this A^0, X^1 fuzzy relation from 1 ton-dimensions is straightforward and doesn't contain any principally new features. It's possible also to regard continuous fuzzy subset X^0 replacing discrete A^0 , but this case is physically uninteresting. The generalization of described formalism to n -dimensions is straightforward and don't include a new features important for us [8]. Formal definition of fuzzy relations is the generalization of regarded examples and can be found elsewhere [8].

3 Fuzzy Mechanics (FM) and Fuzzy States

Usually system quantum state described by Dirac state vector presented as a complex function $\Psi(x)$ or $\varphi(q)$ in arbitrary Q representation. But operationally relevant are a nonnegative distributions $w(x) = |\Psi(x)|^2$ or correspondingly $w^Q(q)$ for other observables. Only them or Q, Q' correlations are observed in the experiments and from them a system state vector Ψ derived. In the real experiment they are realized via statistics of outcome of individual events occurs with probabilities $P(x) = w(x)$, etc.. Note that a quantum state Ψ can be formally expressed via this finite or infinite $\{w^Q\}$ set W^Q covering all possible system observables Q distributions ([13] and ref. therein). This set regarded as the special 'empirical' representation of a quantum states which regarded in our study due to its principal importance despite its practical inconvenience. In some cases a state vector can be restored from the restricted subset of this set. For example a $\text{spin}\frac{1}{2}$ state $\psi(s_z)$ can be restored from $w(s_z), w(s_y), w(s_x)$ despite that the complete set W^Q is infinite.

Now we discuss the transition from Fuzzy geometry to Fuzzy mechanics (FM), analogously to transition from Euclidean Geometry to Classical Mechanics. In such classical case the instant position of the particle is the point in Euclidean 3-space; its physical state corresponds to the point in the phase or configuration Euclidean 6-space. Our main studied system is a massive nonrelativistic particle moving freely or in some potential field. In our model we identify a particle m with the fuzzy point a'_0 regarded in the previous chapter. As follows from described FOSet properties, in fuzzy phase space one can define coordinate axes X, P associated with some material object O chosen as reference frame

(RF) and so m is fuzzy point relative to this ordered axes. Without an evolution in static situation m 'state' supposedly defined by $w(x)$ relative to ordered axis X completely. If the evolution turned on then such object - the fuzzy point m evolves between the configurations where x value is uncertain inside some interval $[x_a(t), x_b(t)]$ as described by $w(x, t)$. Thus m velocity v_x in general can't have certain value and supposedly described by analogous fuzzy parameter with the distribution $w^v(v_x, t)$ on V_x velocity ordered axis associated with O RF. Particle m characterized by its fixed mass parameter m and in this fuzzy X space can move freely or in potential field $U(x)$. For the simplicity at this stage we suppose without proof that a fuzzy momentum p is proportional to the given fuzzy v_x in a sense that $w^v(v_x) = w'(mv_x)$, where w' is p distribution. The principal uncertainty σ_x of m space coordinate x , velocity σ_v and momentum σ_p indicates that FM should have some resemblance with QM and we'll show that m evolution in this theories also has other common features. It supposed below that $w(x), \dots, w^Q(q)$ distributions by means of some experimental procedure can be measured.

We'll suppose that m physical instant state in an arbitrary RF described by a 'fuzzy' state $|g\rangle$ which account all m observables description. For the start we make the minimal assumptions about $|g\rangle$ as mathematical object, to permit the maximum range of possibilities and try to derive its properties from a fuzzy relations in the phase space. $|g\rangle$ set denoted M_s ; it doesn't supposed to be a normalized linear space M_L *a priori*; our aim is to study whether M_s has such properties. Naturally $|g\rangle$ have positive constant norm :

$$N = \|w\| = \int w dx = 1$$

$w(x) = F_x(g)$ is a real positive function on X and some unknown functional of $|g\rangle$. Alike a state in any theory $|g\rangle$ should contain a complete information about an arbitrary m observable Q distribution $w^Q(q)$ and if different Q_i, Q_j related by some constraints or correlations it also should be accounted by g . Remind that in QM x, p distributions are correlated via the commutation relations. $w(x)$ alone can't describe a future g evolution which can depend on $w'(p)$, etc. and it must include the additional components denoted as \bar{g}^x , so that symbolically $g = w \otimes \bar{g}^x$.

Any $|g\rangle$ supposedly can be decomposed into some arbitrary substates i.e. 'states parts', reflecting its fuzzy structure - i.e a simultaneous alternative possibilities coexistence in the phase space. For example consider a fuzzy point m which belongs to the space region E_x which consist of the two noncrossing gaps $Dx_{1,2}$. If $m \in Dx_1$ only, so that its weight in Dx_2 $w_2 = 0$ then $|g\rangle = |g'_1\rangle$. But if both $w_{1,2} \neq 0$ and $w_1 + w_2 = 1$ then $|g\rangle = |g'_1\rangle \oplus |g'_2\rangle$ to which corresponds the following logical proposition for m state :

$$L_E : \quad m \in E_x \cap m \notin Dx_1 \cap m \notin Dx_2$$

We don't define the summation rule *ad hoc* and can't calculate g now, meaning only $|g\rangle = F_s(g'_1, g'_2)$ i.e. g is a general (nonlinear) superposition which stipulates use of \oplus sign. Following the above arguments each g'_i should correspond to at least one 'related' physical state g_i with the norm 1 which describes m confined inside Dx_i with the distribution $w_i(x) = N_i^{-1}w(x)$. We'll regard more formal substates definition and their properties

in Appendix, but here it's enough to use this semiquantitative description. An analogous substates superpositions can be defined like states in the noncrossing intervals $[q_i, q_j]$ for other Q observables. In QM a substates consideration related closely with Ψ definition as a vector in the linear Hilbert space but in FM it results from the m properties as a fuzzy point and has less strictly defined features.

Obviously that for regarded L_E example the further g decomposition is possible - i.e. $g'_i = \sum g'_{ij}$ for any $Dx_{ij} \in Dx_i$ in which Dx_i can be decomposed. Clearly such decomposition can be proceeded to g_{ijk} , etc.. In this approach one can consider the limit $Dx_i \rightarrow 0$ and represent $g = \{w(x); K(x, x', q, q')\}$ where K correlation tensor between different x points and other phase coordinates q . As the example of the minimal FM theory which suits to Fuzzy Geometry picture we regard that m fuzzy state $g_F = \{w(x), K(x, x')\}$ where K correlation tensor doesn't depends of any q and thus $\bar{g}^x = K$ which together with $w(x)$ defines a future $g_F(t)$ state. Later we'll present more arguments in favor of such g structure, but here note only that in general such g_F doesn't admit x -representation and only in the special case discussed below it becomes possible.

We don't formulate yet evolution law for fuzzy states, but at this stage we'll use simple model with the simple assumptions formulated below. The first of them is $w(x)$ norm conservation and corresponding w flow equation should exist. Another one is FM classical limit existence ; when m mass is very large it becomes ordered localized point in phase space and its trajectory defined by Hamilton equations with $H = \frac{P^2}{2m} + U(x)$. Separately should be considered assumption that $|g\rangle$ has x -representation $g(x)$ and $w(x) = F(g(x))$ and thus is local field but we don't assume it at this stage.

$|g\rangle$ describes some extended object evolving in time and to reveal its properties it's instructive to compare it with other extended objects studied in Physics. Of them we'll consider here a classical particle stochastic motion and a classical waves motion. In standard classical statistical mechanics (CSM) particle m initial state is the random point in R^6 and described by probabilistic distribution $P(x, \dot{x})$. Its evolution obeys to Classical Mechanics for an individual trajectory, but their ensemble characterized by probabilistic distribution $P_e(x, \dot{x}, t)$. As CSM state one can use $P(x, \dot{x})$ and $w(x) = P(x)$ is obtained by its tracing. In fact only the quite simple CSM variant regarded here and the real classical statistical theory permits much more complicated options. Until now all assumptions about $|g\rangle$ properties were applicable both for CSM and FM, but now we come to their differences. CSM evolution conserves the state norm and is additive for state components. Additivity (linearity) means that if an initial state is the sum of two states P_1, P_2 with (probabilistic) positive weights r_1^p, r_2^p then each component evolves independently of other component presence the final distribution is:

$$P(x, t) = r_1^c P_1(x, t) + r_2^c P_2(x, t)$$

In distinction as will be shown the pure fuzzy state $|g\rangle$ evolution due to the state structure (source) smearing (SS) effect can violate additivity and is nonlinear for $w(x)$. It constitutes the principal feature of fuzzy evolution and its origin will be discussed here. To illustrate it consider the evolution of m initial state $|g^0\rangle$ at t_0 on X axe (1-dimension) which is the sum of two substates $g_{1,2}$ with $w_{1,2}^0(x)$ disposed in the noncrossing intervals $Dx_{1,2}$. m

Initial state $g^0 = g'_1 \oplus g'_2$ can be regarded as the source $S(g)$ for the produced future state $|g(t)\rangle$ - signal, so that $w_s(x, t) = F_s(g^0, t)$. For the state $g(t)$ the following proposition analogous to L_E describes the fuzzy source structure at t_0 :

$$L_f : \quad S \in (Dx_1 \cup Dx_2) \cap S \notin Dx_1 \cap S \notin Dx_2$$

Suppose that m state evolves freely and so that at some t distributions $w_{1,2}(x, t)$ for the independent $g_{1,2}$ evolution intersects largely. What can be expected for the form of joint distribution $w_s(x, t)$? For CSM states such distribution will be additive sum $w_s^m = w_1 + w_2$, because in any individual event m emitted by Dx_1 or Dx_2 separately and each w_i corresponds strictly to one of this outcomes, i.e. is random source S_R^m . But as follows from L_f in case of fuzzy m source S it can't be attributed to any of Dx_i separately. Due to it w_s form should be such that it's in principle impossible to decompose it into the sum of two components corresponding to Dx_i sources. It should be maximally different from the mixture, so the w_s^m content in w_s should be minimal. Due to it w_s can include principally nonclassical terms alike $\sqrt{w_1 w_2}$ corresponding to m source fuzzy position. Thus SS is the novel nonclassical feature of evolution and will be the main point of our attention throughout this paper.

To study Fuzzy theory we'll use for the comparison also probabilistic mixture $|g^m\rangle$ of several pure fuzzy states $|g_i\rangle$ and for them the additivity is fulfilled in some cases :

$$w^m(x) = \sum N_i w_i^r(x) = \sum w_i(x)$$

where, $N_i = \|w_i\|$; w_i^r, w_i are normalized and unnormalized distributions correspondingly.

We'll start FM study with the simple toy-model of fuzzy evolution and as the example regard the two slits experiment (TSE) often used for a main QM effects discussion. It includes the pointlike source E^m which emits in a wide cone a particles m in the direction of the flat screen with two parallel slits $\Delta x_{1,2}$ divided by the gap $2l_x$ (this set-up regarded to be 2-dimensional). For simplicity E^m supposedly emits the constant m flow and $w(x)$ doesn't depend on t . Due to it our problem becomes analogous to the study of a fuzzy point a'_0 mapping on the 1-dimensional surface X . Behind this screen at distance l_y the photoplate PP_x installed which measures m coordinate x , (normally $l_x \ll l_y$). Δx_i are very small; $\Delta x_i \ll l_x$ and this slits can be regarded as pointlike sources S_i for m state on photoplate. It supposed that the source E^m produces on the screen fuzzy state $|g^{in}\rangle$, which passing through the slits transfers into state $|g^0\rangle$ presented - i.e. $w^0(x) > 0$ only in two separated regions $\Delta x_1, \Delta x_2$ centered around $x_{1,2}^0 = \pm l_x$. Thus $w^0(x) = w_1^0(x) + w_2^0(x)$; $w_1^0 \cap w_2^0 = 0$, corresponding to a fuzzy substates sum : $|g^0\rangle = |g_1^0\rangle \oplus |g_2^0\rangle$.

In our toy-model for TSE case we'll assume that all fuzzy effects depends only on one fuzzy parameter $I_s = \{1, 2\}$ - a number of slit. All other m evolution features supposedly are analogous to CSM. Thus if only one slit Δx_i is open m final distribution will be the same as in particular CSM model. For the simplicity we choose m spread to be spherically symmetric relative to m source - slit S_i with $w_i(x, y) = w_e(\theta)w_r(r)$ where w_e is constant distribution. It results in x distribution on the photoplate :

$$w_i(x) = \frac{\|w_i^0\|}{(x_i^0 - x)^2 + l_y^2} \quad (1)$$

where the source intensity is $N_j = \|w_j^0\|$. Our final qualitative results for the fuzzy smearing effect doesn't depend on exact w_i form, but this ansatz is illustrative because it's the monotonous function without zeroes.

Denote $X_R = \Delta x_1 \cup \Delta x_2$. If m signal is a probabilistic mixture $|g^m\rangle$ of two slits signals its structure described by the proposition for classical random source:

$$L_m : \quad S_R^m \in \Delta x_1 \cup S_R^m \in \Delta x_2$$

from which follows the distribution :

$$w^m(x) = w_1(x) + w_2(x)$$

is the additive sum of the signals from the two slits. For a pure initial fuzzy state from two slits the proposition L_p describes fuzzy source S for TSE analogously to L_f :

$$L_p : \quad S \in X_R \cap S \notin \Delta x_1 \cap S \notin \Delta x_2$$

The distribution $w_s(x)$ form depends on the source properties characterized by L_p from which follows that for a fuzzy source some w_s positive component w_n can't be attributed to $\Delta x_{1,2}$ individually which responds to the regarded SS. Such problems studied in Information Theory and in particular in Images Recognitions topics where fuzzy sets often applied [15, 8]. The first problem is to define SS measure i.e. the separation criteria (SC) for the discrimination of pure fuzzy states and mixture which can be ambiguous. Note that from TSE description SC applied to the situation when the m sources are very small $Dx_i \rightarrow 0$. Also we must find the conditions under which for a pure fuzzy state the complete SS can be achieved. For this purpose let's define the measure of w_1, w_2 overlap $w_1 \cap w_2$:

$$C_x = \int \sqrt{w_1 w_2} dx$$

In principle an alternative measures can be used, but for our problem they lead to effectively same results. If $C_x \rightarrow 0$ TSE mixed and fuzzy distributions simply coincide. Our main observation studied below is that the achievement of maximal SS for w_s turns out to be the severe constraint which in general permit to define the principal m evolution properties.

The input-output flow conservation results in w^m, w_s norm equality :

$$\|w_s\| = \|w^m\| = N_1 + N_2 = \|w_1\| + \|w_2\|$$

, but w_s form should maximally differ from w^m , i.e. w^m content in w_s should be minimal as L_m, L_p propositions indicate. w_s can't depend only on each w_i separately and must include some their nonlinear combinations to become unrelated to any of this slit sources. Formulae for w_s should be applicable also for one open slit resulting in $w_s = w_i$ in this case, thus w_s can be rewritten in the form :

$$\begin{aligned} w_s(x) &= w_1(x) + w_2(x) + w_I(x) \\ w_I(x) &= F_I(w_1^{c_i} * w_2^{d_i}) * F_c(V_g, x) \end{aligned} \tag{2}$$

with $c_i, d_i > 0$; $i = 1, n$. F_I can be decomposed as:

$$F_I = \sum a_i w_1^{c_i} * w_2^{d_i}$$

which can be a finite or infinite sum. If we choose F_c to be dimensionless, and no dimensional parameters contained in F_I , then $c_i + d_i = 1$. V_g denotes all other $|g\rangle$ degrees of freedom (DF) except w_i . We suppose that the interference term (IT) $w_I = F_I * F_c$ admits such factorization, because F_c which describes m normalized distribution on X naturally to be independent of the signal intensities (but not of other $|g\rangle$ parameters V_g).

From $\|w_s\| = \|w^m\|$ it results in $I_g = \int w_I dx = 0$. F_I is symmetric relative to w_1, w_2 and the simplest example is $F_I' \sim 2(w_1 w_2)^{\frac{1}{2}}$. Of course other more complicated F_I should be regarded, but some illustrative calculations will be performed for F_I' . For the comparison note that in CSM $w_I = 0$.

For illustration we start with the most simple and coarse example of SC. w_I can be negative at some x and to consider only nonnegative functions $w_n(x)$ which are nonadditive on w_1, w_2 one should add to w_I some part of $w_1 + w_2$ defined by the condition $\min(w_n) \geq 0$. If to present the signal on the screen as :

$$w_s(x) = k_0(w_1 + w_2) + [(1 - k_0)(w_1 + w_2) + w_I] = k_0 w^m + w_n$$

with arbitrary $k_0 \geq 0$ (it can be also some functions of x as demonstrated below) then it follows that the necessary property of complete SS is $k_0 = 0$. To demonstrate it suppose the opposite: for an arbitrary N_1, N_2 one can apply $k_0 > 0$. Then w_s is the sum of two distributions w^m and w_n with the norm $k_0, 1 - k_0$. In this case w_s distribution corresponds to the probabilistic mixture of two ensembles with distributions w^m, w_n . But this contradicts to L_p proposition which exclude any presence of $|g^m\rangle$ which characterized by L_m and produce w^m . Now let's find how this fact restricts w_s form. For the simplicity suppose that $w_{1,2} > 0$ for all x alike (1) and so $w^m > 0$. Then to exclude w^m admixture in w_s it's enough to demand that at least in one point x_0 in which $w^m(x_0) > 0$ one have $w_s(x_0) = 0$. It seems a very important w_s property which shows that m fuzziness can deform $w^m(x)$ form significantly and reveals in fact FM nonlocal, nonlinear features. Below more strict w_s properties will be found and in particular it will be shown that w_s should oscillate around w^m and have many poles x_j .

After this simple example let's regard more subtle SS criteria - SC for w_s which will be used throughout our formalism. As in the first case let's decompose w_s into the additive and nonadditive parts :

$$w_s(x) = k_{w1}(x)w_1(x) + k_{w2}(x)w_2(x) + w_n(x)$$

where k_{wi} are an arbitrary positive functions and w_n defined via w_I, w_i analogously to above example. In this case we demand again $\min(w_n) \geq 0$ for any x characterizing w^m admixture of any form. Obviously the largest SS - sources smearing achieved for the approximately equal signal intensities: $N_{1,2} = \frac{1}{2} \pm \epsilon$ at $\epsilon \rightarrow 0$ where F_I of (2) has its maximal value. From that SC can be formulated as follows : in this limit in w_s decomposition one should get $k_{wi}(x) \rightarrow 0$, so admixture of additive S_i signals is negligible.

We argue that such SC corresponding to maximal SS for w_s stipulated by two interrelated conditions. The first necessary condition is the complete $w_{1,2}$ overlap $C_x = 1$ which means $w_1 = w_2$. The second condition assumes that in FM for any $w_i(x)$ the corresponding SS class of functions F_c exists which permit to achieve maximal SS. To demonstrate the first condition meaning note first that if in some interval Dx $w_2(x) = 0$ and $w_1(x) > 0$ (or vice versa) then in this Dx interval the signal from one slit in $w_s(x)$ presented only i.e. $k_{w1}(x) = 1$ and $w_n(x) = 0$ which excludes maximal SS achievement. If $w_1 > w_2$ (or vice versa) in some Dx then w_s in this bin is also unambiguously affected but its proof is a bit more complicated. For example for F'_I one obtains:

$$k_{w1,2} = 1 - \frac{2\sqrt{w_1 w_2}}{w_1 + w_2}$$

and $k_{wi} > 0$ in any x where $w_1 \neq w_2$ which contradict to maximal SS. It's easy to check that $w_1(x) = w_2(x)$ for TSE can be achieved at $l_y \rightarrow \infty$, when $w_i = \text{const}(x)$. In general to find this SS F_c class is quite complicated problem, but for our study it's enough to consider several simple cases.

From w_s Fourier transform analysis it follows that for $w_{1,2} \rightarrow \text{const}(x)$ independently of F_I this SS class of functions is $F_c(x) = \cos(p_f x + \varphi_f)$ with an arbitrary parameters p_f, φ_f . Really in this w_i limit for any F_I it results in $F_I \rightarrow \text{const}(x)$ also and for such F_c it follows $\int F_I F_c dx \rightarrow 0$ which is the normalization condition on w_s of (2) oscillations. Also this oscillations have the maximal amplitude compatible with $w_s \geq 0$ and $w_s(x_i) = 0$ for the infinite number of x_i poles if $F_I = 1$ under condition $w_1 = w_2$. w_s dependence of p_f, φ_f evidences that K or \bar{g}^x components of g are necessary in addition to $w(x)$ for the unique final state description. It seems a quite important observation that FM results in oscillating w_s distribution in TSE, because in general it demonstrates the possibility of fuzzy $w(x)$ nonlinear evolution. Remind that experimentally such oscillations observed in TSE which is one the direct QM nonclassicality demonstrations.

Of course to be precise one should study w^0 smearing over all space (X,Y in our case) at given t but $w_s(x)$ at given large l_y permits to achieve the maximal possible SS and reproduces the main features necessary for the comparison of our toy-model with QM. Despite SC was regarded in toy-model with one fuzzy parameter all this SC formalism for pure and fuzzy states smearing are applicable for arbitrary FM theory and will be used below with account of possible w_i time dependence.

We must stress that this SS - i.e. the principal smearing of initial state is the universal FM feature and in general case of arbitrary state evolution the final $w_s(x, t)$ has analogous dependence on w_0 with additional account of possible g time dependence. TSE is just the illustrative example which reveals SS features in the most simple form. In particular w_s ansatz (2) is applicable for a wide class of nonlinear theories.

4 Evolution in Fuzzy Mechanics

Now we try to develop our approach to m state evolution in a more formal way and try to find FM evolution properties of free m motion in 1-dimension which follows from

the maximal SS conditions regarded in the previous chapter. Note that below in our calculations a generalized complex and real functions, alike Dirac $\delta(x)$ are used, but their physical meaning is well understood in standard QM and the same approach to them applied here [14].

Consider that experimentalist prepares in 1-dimension at t_0 an arbitrary m pure fuzzy state $|g_0\rangle$ with $w_0(x)$ distribution localized inside x_a, x_b interval; other g_0 parameters are unimportant. After that m evolves freely to some unknown $g(t)$ with $w_s(x, t)$. As was argued above due to SS - fuzzy smearing $w_s(x, t)$ have such form that at any t it's impossible to relate $w_s(x_i, t)$ origin to a pointlike sources in arbitrary small bin Dx_j at t_0 . As the result $w_s(x, t)$ has nonlinear w_0 dependence which calculated here. We'll start with considering initial m pure fuzzy state $g_0 = g_1^0 \oplus g_2^0$ at t_0 localized in two such small bins i.e. $n_s = 2$:

$$w_0 = w_1^0 + w_2^0 = N_1\delta(x - x_1) + N_2\delta(x - x_2) \quad (3)$$

with an arbitrary x_1, x_2 and $N_1, N_2 > 0$; $N_1 + N_2 = 1$. This set-up is in fact 1-dimensional TSE analog and for it w_s ansatz (2) and correspondingly first SS condition $w_1 = w_2$ should hold also at any t independently of $|x_1 - x_2|$ value. From x invariance it follows that for each individual source $w_i(x) = f_w(t)const(x)$; the time factor f_w has no principal meaning [17]. Really for each realistic m source S_i $w_i(x)$ centered around x_i (or correlated with it) so as the result for any pair of them the overlap $C_x < 1$ and only $w_i(x) = const$ avoids it giving $C_x = 1$. Remind that for TSE such w_i achieved in the limit $l_y \rightarrow \infty$. Yet as was shown above for the superposition of two substates $g_{1,2}$ with such constant space distributions the corresponding F_c class resulting in maximal SS is :

$$F_c(x, t) = \cos \beta_{12} = \cos[p_{12}^f(t)x + \varphi_{12}(t)]$$

with an arbitrary functions p_{12}^f, φ_{12} . w_s ansatz (2) is applicable here also and in its terms:

$$w_s(x, t) = w_1(x, t) + w_2(x, t) + w_{12}(x, t) = f_w(t)[(N_1 + N_2) + F_I(N_1, N_2)F_c(x, t)] \quad (4)$$

We don't obtain at this stage F_I , except its agreement with (2) but it will be calculated below. The only its established feature is its maximum value $F_I = 1$ at $\|w_i\| = N_i = \frac{1}{2}$ which follows from SC conditions and in particular it gives periodical $w_s(x_i^f) = 0$. Note that this SC are correct for the pointlike sources with the width $Dx \rightarrow 0$ but aren't applicable for the sources of arbitrary width.

Let's discuss the obtained intermediate results. First, up to undefined at this stage F_I, k_f, φ this ansatz for $w(x, t)$ coincide with the corresponding QM path integral calculations and it doesn't seems to be just occidental coincidence [17]. As was mentioned above the meaning of $\delta(x)$ and $const(x)$ distributions formulated in QM formalism as the spectral decomposition of standard L^2 functions [14] and in our theory it will be used in the same sense. Despite, the physical meaning of this statements will be analyzed in some limit below. For example one can regard results for $\delta(x)$ sources as the limit for the source S_G of the initial gaussian form $w \sim e^{-\frac{x^2}{\sigma^2}}$ for the limit $\sigma \rightarrow 0$. When one can claim that in this limit w_s smearing of two such sources tends to be complete in the final state at any t . Analogously the constant w distribution in S_G final state can be approximated as

gaussian with infinitely large dispersion $\sigma \rightarrow \infty$ Hence all the following results should be regarded as the analogous asymptotic propositions which are exact in the suitable limit. Eventually if φ, p^f, f_w, F_I will be found and the consistency of such theory proved one can obtain via the spectral decomposition an evolution of any initial g^0 and the corresponding $w^0(x)$. To derive this functions let's regard the same set-up for three analogous pointlike sources $n_s = 3$ at t_0 to which corresponds the general w_s ansatz :

$$w_s(x, t) = \sum w_i(x, t) + \sum_{j=1}^3 \sum_{i < j}^3 w_{ij}(x, t) + w_{1,2,3}(x, t) \quad (5)$$

which is the generalization of (4) with the new triple term $w_{1,2,3}$ which can depend on all three w_i . Yet SC ansatz for $n_s = 2$ should be fulfilled for each pair of sources i, j (2, 3 for example) resulting in their maximal SS independently of other sources presence; otherwise fuzzy smearing will become incomplete for $n_s \geq 3$. $w_{1,2,3}$ presence should deform this ansatz with the rate depending on $w_3(x)$. From that we conclude that $w_{1,2,3} = 0$ and any w_{ij} ansatz coincides with w_{12} of (4) (here and below $i < j$ in double indexes). The same arguments are valid for an arbitrary $n_s \geq 3$ which permit to generalize in the obvious way w_s for $n_s = 3$. Such w_{ij} can be negative at some x and thus for $n_s \geq 3$ without the additional constraints on $p_{ij}^f, \varphi_{ij}, F_I$ one finds that w_s can become negative which is nonsense. It's argued here that the only solution for w_s which is equivalent to this constraints is :

$$w_s(x, t) = \left| \sum_{i=1}^{n_s} \Phi_i(x, t) \right|^2 \quad (6)$$

where Φ_i are some N_i dependent complex functions. We omit the detailed proof which is quite elementary but tedious and indicate only its main points. Let's denote as β constraint the equality $|\sum \beta_{ij}(x) - 2n\pi| = 0$. As the first step we find w_s for $N_1 = N_2 = N_3$ fixing F_I and consider only p_{ij}^f, φ_{ij} variations. It follows that for an arbitrary p^f, φ even the infinitesimal $\delta p_{ij}^f, \delta \varphi_{ij}$ variations results in $w_s(x) < 0$ in some x interval if due to this variations β constraint violated. From the comparison with w_s of (5) β constraint for equal N_i results in :

$$w_s = \left| \sum_{l=1}^{n_s} \sqrt{N_l} e^{i\gamma_l(x, t)} \right|^2$$

with $\beta_{ij} = \gamma_i - \gamma_j$. From β constraint and FM x coordinate invariance γ_l can be derived up to $2n\pi$:

$$\gamma_l(x, t) = \eta(t)(x - x_l)^2 + \alpha_l(t)$$

where η, α_l are an arbitrary real functions. After that due to F_I, F_c factorization in (2) we can use the obtained relations and regard F_I parameters variations. After the analogous calculations one obtains $F_I(N_i, N_j) = 2\sqrt{N_i N_j}$ and all this results can be extended on arbitrary n_s copiously.

The simplest consistent $|g\rangle$ representation in this case is a complex function $g(x, t) = \sum \Phi_l(x, t)$ which corresponds to $w_s(x, t)$ of (3). If an initial state for $n_s = 2$ to

rewrite also via $\Phi_l(x, t_0)$ one obtains :

$$g_0 = \sum_{n_s} \sqrt{N_l} \delta(x - x_l) e^{i\alpha_l^0}$$

where $\alpha_l^0 = \alpha_l(t)$ are an arbitrary real constants. Also it gives:

$$\eta(t) = \frac{im}{2(t - t_0)}; \quad f_w(t) = \frac{m}{2\pi(t - t_0)}$$

This ansatz can be extended on arbitrary n_s and from this relations p^f, φ, γ easily restored; in particular:

$$p_{ij}^f(t) = -i\eta(t)(x_i - x_j)$$

Thus an n_s initial state besides N_l depends only on the $n_s - 1$ parameters $\alpha_l^0 - \alpha_i^0$ which defines the correlations between the fuzzy sources. If we admit that for g_0 one can transfer from the sum of $n_s \rightarrow \infty$ to an integral on x smoothly then it describes the spectral decomposition of an arbitrary complex $g_0(x) = \sqrt{w_0(x)} e^{i\alpha(x)}$ and its evolution for a free m state :

$$g(x', t) = \int G(x' - x, t - t_0) g_0(x) dx = \int e^{\frac{im(x' - x)^2}{2(t - t_0)}} g_0(x) dx$$

which coincides with the free m evolution in QM path integral formalism and G is free Feynman propagator [17]. Summing up we conclude that the obtained ansatz coincides with QM for the free m evolution and in particular reveals FM state g evolution linearity. $|g\rangle$ is the normalized vector (ray) of complex Hilbert space \mathcal{H} which corresponds to our set M_s .

Now we need to formulate Hamilton formalism for FM. For complex $g(x)$ from x invariance of free m motion the momentum can be only the Hermitian operator $p = i\frac{\partial}{\partial x}$ with $[\hat{p}, x] = i$, so that $\bar{p} = \frac{\partial \alpha}{\partial x}$ [3]. Yet we know that the linear complex functions evolution described by Schrodinger equation (SE) of QM, where Hamiltonian H becomes Hermitian operator. It guarantees also m flow conservation and restores classical limit for arbitrary H [3]. From the above arguments the free Hamiltonian follows $\hat{H}_0 = \frac{\hat{p}^2}{2m}$ and the natural FM generalization for the m potential interactions is : $\hat{H} = \hat{H}_0 + U(x)$.

It's interesting to check if the maximal SS condition conserved in the presence of interactions. As the example we regard the harmonic oscillator $U = \frac{\omega_0^2 x^2}{2}$ and consider the evolution of the initial pure state g_0 consist of two pointlike sources. This state at $t > t_0$ calculated by path integral formalism. We omit here the detailed results which can be reproduced easily [17] and just notice that w_s in this case coincides with the free w_s of (4) but with the different $f_w(t), p^f(t), \varphi(t)$. It evidences that maximal SS principle conserves its validity even in the strong potential field but to prove it finally needs more calculations for an arbitrary U .

The correlation tenzor of minimal FM model corresponds to a quantum phases differences $K(x, x') = \alpha(x) - \alpha(x')$ and quantum phase properties obtains quite natural explanation : the real physical parameter is $K(x, x')$ - the fuzzy correlation between x, x' and α is its x representation ambiguous up to $2n\pi$.

The same results as presented in this chapter can be obtained from assuming some weak M_s set properties phenomenologically. Initially g was regarded as the abstract mathematical object and it don't assumed *a priori* that g corresponds to any linear array (vector) of functions on phase space. We assume only that the substates summation is associative and the substates has the selfsimilarity property. The calculations details can be found in Appendix.

From the demonstrated equivalence FM and standard QM the fuzzy smearing can be described in the spectral decomposition framework. Really, consider an arbitrary QM initial state $\psi_0(x)$ at t_0 with $w_0(x) > 0$ in some E_x interval. Let's select its arbitrary substates $\psi'_{1,2} - S_i$ sources localized in the small bins $Dx_{1,2} \in E_x$ such that $\psi'_i(x) = \psi_0(x)$ if $x \in Dx_i$, $\psi'_i(x) = 0$ otherwise. From the standard QM calculations follows that for QM free evolution for this S_i at any t in the limit $Dx_i \rightarrow 0$ maximal SS achieved for $w(x, t)$. Thus considered FM results for the pointlike sources acquires the consistent asymptotic meaning via QM states spactral decomposition. The same results can be obtained for QM substates of gaussian form w_G in the limit $\sigma \rightarrow 0$ which were discussed above for FM states.

For the illustration of FM properties it seems interesting to discuss how m momentum distribution can be derived from Fuzzy Geometry directly without use of p operator definition obtained form FM above. As was mentioned already it supposed that in FM relation $p = mv_x$ in one dimension is true in a sense that $w^v(v_x) = w'(mv_x)$ and velocity defined by relation: $\bar{v}_x = \frac{\partial \bar{x}}{\partial t}$. Consider again the state $|g\rangle$ which can be presented as sum of two substates $|g_{1,2}\rangle$ which don't cross on x axe : $w(x) = w_1(x) + w_2(x)$; $w_1 \cap w_2 = 0$, but their independent momentum distributions w'_1, w'_2 cross on p axe. Considering relation between w and w' in our approach w can be formally defined as the 'source' of w' with the relation for each w_i $w'_i(p) = N_i w_i^{rp}(p)$ where w^{rp} is the normalized p distribution. If for example this is the mixture $|g^m\rangle$ it gives : $w'_s = w'_1 + w'_2$ is additive on $\|N_i\|$ analogously to CSM and don't result in any constraint between w, w' . For a pure fuzzy $|g\rangle$ from L_p and the arguments given above :

$$\begin{aligned} w'_s(p) &= w'_1(p) + w'_2(p) + w'_I(p) \\ w'_I &= 2N_1^{\frac{1}{2}} * N_2^{\frac{1}{2}} * F'_c(V_g, p) \end{aligned} \quad (7)$$

where V_g are other $g_{1,2}$ parameters. Let's consider again the regarded 1-dimensional set-up with two pointlike sources with the initial x distribution (3). Suppose that an experimentalist measures only p (or v_x) at t_0 without acquire any x information. In QM it's permitted by its axioms but in FM one should be careful with such assumption. Then $w'_s(p)$ should have such nonadditive form that it should be impossible to ascribe its components to the particular source. From the analogous chain of arguments as for $w_s(x)$ but now applied for the same time t_0 it follows that for the single source $w'_i(p) = \text{const}(p)$ and

$$w'_s(p) = f_w^p[N_1 + N_2 + 2\sqrt{N_1 N_2 \cos(d_p p + \varphi_p)}]$$

with the arbitrary f_w^p, d_p, φ_p . Thus QM momentum distribution derived without addressing to p operator ansatz. For the complex $g(x)$ it corresponds to commutation relation:

$[\hat{p}x - x\hat{p}]g = ig$ which leads to Heisenberg uncertainty relation $\sigma_x\sigma_p \geq 1$ for w, w' which can be regarded as the weak (inequality) constraint. This consideration seems to us important because it indicates that the origin of the commutation relations lays in Fiset properties of the phase space.

Thus QM corresponds to FM and $g(x)$ corresponds to $\Psi(x)$ Dirac state vector in x representation. Such theory is the simplest nonlinear dynamical theory with 'square root' w^Q nonlinearity, which expressed via linear complex $g(x)$ evolution. It's important to note that SE only isn't enough and one needs to define the relation between $|g\rangle$ and w^Q distributions to construct the complete physical theory. Extension of this FM theory on 3 dimensions is straightforward and we omit its consideration here.

5 Discussion

In this paper we've demonstrated that QM and mechanics on fuzzy manifold - FM has the close correspondence at least in 1-dimensional case. Proposed FM model contains some phenomenological statements which is inevitable for the theory dealing with continuous spectra. Our theory by no means aimed to disprove QM in its contemporary form, mainly we try to develop adequate language to describe the quantum phenomena, and for that purpose Fuzzy Geometry seems appropriate. Obtained results indicate that FM predict effects analogous to QM interference induced by fuzzy smearing or SS - sources indistinguishability which is the principal property of fuzzy states absent in Classical Theory.

Note that QM in this approach seems has some formal analogy with classical constrained dynamics [5]. In such theory n_e - number of effective dynamical coordinates q' equal to number n_m of formal coordinates q minus number n_c of theory dynamical constraints. In CSM with constraints its formal state $P(q, p)$ expressed by means of some linear operators via effective state $P'(q', p')$ with less number of degrees of freedom (DF). Analogously to it w^Q set in FM reduced to single complex function $g(x)$ due to 'fuzzy constraints'. Of course this is just distant analogy and QM has more intricate structure, but it illustrates some its properties. In FM constraints are weaker then in classical constrained dynamics, so that no DF can be removed. Mainly they connect observables related to the same DF alike x, p resulting in constraints inequalities of the kind $\sigma_x\sigma_p \geq 1$. Proposed FM formalism pretend to explain the nature of QM commutation relations as originating from SS property of evolution on fuzzy manifold.

In fact the need to use complex Hilbert space for quantum states results from observation of m interference nonlinear on initial $w^0(x)$. It can't be principally described by linear positive states alike CSM. In FM approach QM state vector $|\psi\rangle$ meaning as the mathematical but unphysical object demonstrated. The true physical entities are w^Q distributions observed experimentally. $|\psi\rangle$ is the mathematical tool which simplify calculations of w^Q evolution from initial to final distributions which difficult to perform directly due to their essential nonlinearity. Note that we used $\hbar = 1$ Planck constant calibration throughout our formalism as it's done in relativistic QM. But it's clear that this Planck constant simply connect x, p scales of phase space and has no separate meaning.

Standard QM postulates that particle state described by Dirac state vector in Hilbert

space with corresponding evolution equation. From that it derives uncertainty of x, p and m paths, their interference, etc. On the opposite Fuzzy theory admits that x coordinate and consequently m path are principally fuzzy and from this axioms we attempt to prove that m states set M_s is complex Hilbert space with Schrodinger evolution formalism. Standard Schrodinger quantization substitutes classical pointlike particle by the new object - Dirac state vector conserving classical space-time structure. FM in fact takes the opposite approach : particle stays to be material 'fuzzy' point, but space-time set structure changed to Foset.

Note that one of unsolved problem in this FM formalism is the nature of physical RF space coordinates - i.e. 'target space' relative to which we describes our states $g(x)$. We associate RF with some massive object which interactions with other objects - solid rods, photon bunches defines geometric ordered points x_i .

Clearly our FM approach has the close relation to von Neuman algebra and Quantum Logics, especially in its fuzzy sets - i.e multivalued logics formulation [10]. In this theories the set structure of phase space is ordered one, but Logics of propositions for this space is multivalued with at least 3 outcomes : $Y/N/U$. In our theory standard Boolean logics used, but this multivaluedness in fact transferred into phase space Foset and implemented inside its geometry. Yet our approach seems to be closer to physics, which always use some geometry for description of objects relations and General Relativity is good illustration of this thesis.

FM ideas seems to be related to Quantum reference Frames theory developed by Aharonov and Kaufherr. They have shown that in nonrelativistic Quantum Mechanics (QM) the correct definition of physical reference frame (RF) must differ from commonly accepted one, which in fact was transferred copiously from Classical Physics [1]. The main reason is that to perform exact quantum description one should account the quantum properties not only of studied object, but also RF, despite the possible practical smallness [18]. The most simple of this RF properties is the existence of Schrodinger wave packet of free macroscopic object [3]. Remind that physical RF F^0 is normally associated with some macroscopic object M which can perform measurements of studied particles, for example it can be satellite in outer space. M is binded system of atoms each of them obeys to QM laws - i.e. evolves according to Schrodinger equation (SE). It follows then that F^0 c.m. motion also obeys to SE relative to any other RF - F^1 and F^0 quantum state will be localized wave packet with dispersion σ_x [3]. It introduces additional uncertainty into the measurement of object space coordinates in F^0 . Furthermore this effect results in the states transformations between two such RFs which includes quantum corrections to the standard Galilean group transformations [1]. In their work Aharonov and Kaufherr formulated Quantum Equivalence Principle (QEP) in nonrelativistic QM - all the laws of Physics are invariant under transformations between both classic and this finite mass RFs which called quantum RFs. QRF theory if to take it seriously prompts to reconsider QM foundations related to space-time structure. Really QRF existence suppose that space coordinate is the subjective entity connected with the object regarded as RF.

Let's regard briefly some possible consequences of QM equivalence with regarded Mechanics on fuzzy manifold - FM. First, one can use space-time Foset structure at quantum level, which possible can be applied in some specific problem, alike Gravity quantization.

Here we studied the fuzzy phase space structure, but in relativistic case also space-time structure must be modified analogously. It will be interesting to compare it with Non-commutative Geometry approach to space-time at small scale [4].

A Appendix: Fuzzy states Decomposition

Here we regard the complementary derivation of FM results from the assumption of some simple fuzzy states properties. Note that initially g was regarded as the abstract mathematical object which properties must be found and we don't assume *a priori* that g corresponds to any linear array (vector) of functions on phase space.

We define formally that $|g'\rangle$ is substate, i.e. the object which can be operated analogously to $|g\rangle$, but in general don't describe m state completely. $|g'\rangle$ is linear in M_s if it can be presented as $|g'\rangle = r|g^0\rangle$ for $0 \leq r \leq 1$; for some arbitrary state $|g^0\rangle \in M_s$, but this substates linearity is too strong condition which means in fact $g \in M_L$ and isn't necessary. The necessary feature is the weaker condition : $|g'_1\rangle$ norm $N'_g \leq 1$ and for any Q for $|g'_1\rangle, |g^0\rangle$ we have : $w^Q_1(q) = N'_g w^Q(q)$; for strong condition it corresponds to $N'_g = f_N(r)$. We'll call two substates $g'_{1,2}$ parallel if for them any $w^Q_{1,2}$ differ only for their norms. Thus g'_i is parallel to a separated state g_i . We'll regard more formal substates definition below, but here it's enough to use this semiquantitative description. An analogous substates superpositions can be defined like states in noncrossing intervals $[q_i, q_j]$ for other Q observables. In QM a substates introduction related closely with Ψ definition as a vector in the linear Hilbert space but in FM it results from the from m properties as a fuzzy point. We assume also that substates summation is associative :

$$(g_1 \oplus g_2) \oplus g_3 = g_1 \oplus (g_2 \oplus g_3)$$

For illustration we'll suppose first also substates linearity, but afterwards perform the same calculations dropping this assumption conserving only the more weak property of states selfsimilarity.

Regarded TSE toy-model features supposedly demonstrates that particles m states evolution is nonlinear for $w(x, t)$ and for others w^Q . From TSE analysis FM formal analogy with waves evolution is straightforward : in both cases one has the flow conservation and strong interference effects and this analogy should be explored in detail. Due to well-known problems with nonlinear evolution the wave theory results prompts us to look for the special $|g\rangle$ representation which obeys to linear evolution equations and reproduces w^Q by some nonlinear relations $w^Q(q, t) = F^Q(g, q, t)$. In this linear case for TSE follows that the state on the photoplate presented as $|g_f\rangle = |g_{1f}\rangle \oplus |g_{2f}\rangle$ where g_{if} produced by g_i and $w_s = F(g_f)$.

Now suppose that a fuzzy states admits selfsimilar decomposition. It means that any fuzzy state $|g\rangle$ can be decomposed identically into the system of parallel to g substates $|g_i\rangle$. Moreover any such substate in its turn can be decomposed into another substates system. For the start here we'll assume parallel substates linearity - i.e. strong condition formulated above, but after that calculate it for weak condition which don't suppose g

linearity. Consider arbitrary fuzzy state $|g\rangle$, obviously it can be rewritten as :

$$|g\rangle = |g_1\rangle \oplus |g_2\rangle = s_1|g\rangle \oplus s_2|g\rangle$$

with $0 \leq s_{1,2}$ and $s_1 + s_2 = 1$. This equality supposedly fulfilled for our undefined $|g\rangle$ summation rule, whatever it is. Note that such equality holds for quantum state vectors, yet calculating corresponding QM probabilities one should formally account appearing IT, which is the essence of the regarded ansatz. For parallel substates $w_i(x) = f(s_i)w(x)$ for any monotonous f , so we can also use f value to identify substates. Interference of parallel substates obviously can be only constructive and independent of x , so $F_c(x) = c_g \geq 0$. For the lack of place we don't prove it here, but following calculations illustrate it. Let's rewrite (2) for this substates and obtain from it by integration over x the relation for their norms. If to denote $u_0 = 1; u_1 = s_1; u_2 = s_2$ and remind that $f(1) = 1$ after canceling $\|w_s\|$ on left and right side one obtains :

$$f(u_0) = f(u_1) + f(u_2) + F_I(f(u_1), f(u_2))F_c \quad (8)$$

This equality is supposedly true also for substates and if we decompose $|g_2\rangle$ analogously :

$$|g_2\rangle = |g_3\rangle \oplus |g_4\rangle = s_3|g_2\rangle \oplus s_4|g_2\rangle$$

with $s_{3,4} \geq 0; s_3 + s_4 = s_2$ then equality (8) must be true for substitution $u_0 = s_2; u_1 = s_3; u_2 = s_4$. This equalities must be hold simultaneously for arbitrary s_i inside described intervals and constitute equations system which permit to derive f, F_I . Assuming that no new dimensional parameters appears, after simple algebra omitted here obtain:

$$f(z) = z^n; \quad c_g = 1; \quad F_I(z_1, z_2) = (z_1 + z_2)^n - z_1^n - z_2^n; \quad n = 1, 2, \dots \infty$$

Now we drop a substates linearity assumption changing it to weak condition for norm and use as substate $|g_2\rangle$ parameter $f = f(s_2)$, so admit $w_2 = fw$; $w_1 = f'(f)w$ and $f + f'(f) \leq 1$. From it and (8) we obtain equality:

$$1 = f + f'(f) + F_I(f, f')F_c$$

Then again decomposing $|g_2\rangle$ into $|g_{3,4}\rangle$ from corresponding equalities system the solution obtained :

$$\begin{aligned} f' &= (1 - f^{\frac{1}{n}})^n; \quad c_g = 1; \\ F_I &= (f^{\frac{1}{n}} + (f')^{\frac{1}{n}})^n - f - f'; \quad n = 1, 2, \dots \infty \end{aligned} \quad (9)$$

So even without substates linearity we get essentially the same results which restricts F_I choice severely.

Here $n = 1$ corresponds to classical probability without IT i.e. to CSM. $n = 2$ possibly corresponds to QM and classical waves interference and so deserves special attention. Note that $F_c = c_g$ only for parallel substates, but it can be also negative for destructive interference and so for arbitrary substates $-1 \leq F_c \leq 1$. Yet for $n \geq 3$ for $F_c \leq 0$ w_s

becomes negative which principally contradicts to its definition. In general this solutions exists also if n changed to rational $\frac{j}{l}$ or continuous y_n , but it doesn't bring something new in our arguments and we'll regard only natural n .

Yet by use of associativity even for weak condition without g linearity by renormalization of given states one can obtain new states g^r which obeys to linearity. For example for $n = 2$ one can substitute $s'_1 = \sqrt{f}$; $s'_2 = \sqrt{f'}$ and it follows $s'_1 + s'_2 = 1$. Thus we can express $|g_i^r\rangle = s'_i|g\rangle$ and in this case strong and weak conditions on substates coincide, below we regard g and g^r states on the same ground.

Until now we operated only with real and not complex values which will be regarded now. Consider for $n = 2$ and $|g_\alpha\rangle = e^{i\alpha}|g\rangle$ for real α if to settle $f = |s|^2$ for complex s then $|g_\alpha\rangle$ describes the same distribution $w(x)$. Really any w^Q depends on g internal structure and can't change after such multiplication. Let's return to $|g\rangle$ selfsimilar decomposition, but permit s_1, s_2 to be complex : $s_i = d_i e^{i\alpha_i}$; $d_i \geq 0$. Conserving $s_1 + s_2 = 1$ constraint now for $u_0 = 1$; $u_i = s_i$ eq. (8) fulfilled with f, F_I defined above. Now we can't claim $F_c = 1$ because interference of substates with complex s_i can't be guaranteed to be constructive. From simple calculations if to settle that the complex component of (8) equal to 0 one obtains that $F_c = \cos(\alpha_1 - \alpha_2)$

Obtained results deserves some comments. In standard QM such calculus don't play any role, because IT form defined by Hilbert space properties introduced *ad hoc*, in particular the scalar product definition. In our model we defined only minimal M_s properties, and thus IT form is arbitrary and is derived or at least constrained assuming M_s fuzzy states selfsimilar structure. The selfsimilarity means that the fuzzy states introduced consistently and in particular any substate g_i has all the formal properties of complete m state. We can't exclude completely the theories with $n \geq 3$ now, but at least they seems to include quite intricated nonlinear effects. Such selfsimilarity is quite often meet in other theories and so can have universal meaning. Note that the nonunique norm and distance definition permitted for complex functions spaces M_L^P for $P \geq 1$ [16], but only for $P = 2$ scalar product can be defined and M_L^P becomes Hilbert space. Our calculations leads to the analogous result : $n = 2$ is preferable for consistent theory , but from different arguments and this correspondence deserves further study. Anyway, our ansatz at $n = 2$ has $|g\rangle$ scalar product definition.

To get this results we admitted only $|g\rangle$ state decomposition into parallel substates which observed properties identical to $|g\rangle$ except the norm. Earlier we assumed that $|g\rangle = \sum \oplus |g_i\rangle$ sum of substates in noncrossing regions Dx_i on which effective X surface can be decomposed. In the limit of $Dx_i \rightarrow 0$ it can be expressed as $|g\rangle = \int \oplus |g_x\rangle dx$. Now we regard more restrictive hypothesis that $|g\rangle$ has x -representation i.e. fuzzy state described completely via $\vec{g}_x = \{g^\mu(x)\}$; $\mu = 1, n$; countable set (vector) with arbitrary n number of real or complex functions components, plus scalar product definition. Consequently summation rule \oplus describes component sum and states set is M_L linear functions space.

This x representation corresponds well to fuzzy geometry which supposes that m evolution features can be encoded in some functions on X axe. Turning back to $g_F = \{w, K(x, x')\}$ structure regarded in chap.3 note that it admits x -representation if K can be presented as $K = K_F(x) - K_F(x')$ which will be obtained here eventually.

Now we can apply the selfsimilar decomposition in any point x and s_i can be arbitrary

functions. Eq. (8) now is applicable in any point x for $w_s(x)$ and $s_i(x)$ with the same constraint $s_1 + s_2 = 1$. It follows for complex $s_i(x) = d_i(x)e^{i\alpha_i(x)}$ for arbitrary real functions d_i, α_i that it should be

$$F_c(x) = \cos(\alpha_1(x) - \alpha_2(x))$$

Thus if it's possible to factorize from $\vec{g}_{1,2}(x)$ states the complex multipliers $g_{1,2}^c(x)$ than for two states sum $g_1 + g_2$, F_c depends on them along with this ansatz for $s_i(x)$ and $F_I = 2\sqrt{w_1 w_2}$ for w_s .

After this formalism development the general evolution problem (GEP) can be regarded: experimentalist prepares arbitrary fuzzy state g_0 and should calculate the future state $g(t)$. g_0 can be presented as the sum of sources w_i^0 in small Δx_i : $|g_0\rangle = \sum |g_i\rangle$. Alike for TSE where it follows from L_p property in general pure case m source doesn't attributed to any Δx_i : $m \notin \Delta x_i$. Due to it g evolution should smear the signals from different Δx_i for final $w_s(x, t)$ which should obey to SC condition and so restricts possible g evolution. From the arguments discussed for TSE :

$$w_s(x, t) = \sum w_j(x, t) + w_I(w_j^0, x, t)$$

From selfsimilarity arguments it follows for IT :

$$w_I = \sum \sum \sqrt{w_i^0 w_j^0} G(x, t, x_i, x_j, g_i^p, g_j^p)$$

where G is double correlated m propagator and g_i^p are g_i parameters different from w_i^0 . Note that in distinction from TSE the time dependence of w_s becomes important. Of course other w^Q distributions has the same principal properties, but for x they are most obvious.

It's easy to show that single real function $g_r(x)$ can't satisfy to formulated demands for g_x . If we regard the complex function $g(x) = d(x)e^{i\alpha(x)}$ with $d \geq 0$ and α - real then it follows $w(x) = g^*(x)g(x)$. For TSE $w_I = 2d_1 d_2 \cos \alpha_{12}$; where $\alpha_{12}(x) = \alpha_1(x) - \alpha_2(x)$. Thus if for TSE model dynamics results in monotonous $\alpha_{12}(x)$ and fuzzy separation criteria SC fulfilled for oscillating w_s then complex $g(x)$ can be good candidate for fuzzy state. Yet we know that such dynamics supplied by Schrodinger equation (SE) of QM, where Hamiltonian H becomes Hermitian operator. It guarantees also m flow conservation and restores classical limit for arbitrary H [3]. The same is true for GEP $g(t)$ and it can be shown that fuzzy SC fulfilled for arbitrary initial $g(x)$ due to quantum interference. $w_s(x, t)$ has the form analogous to (2) but with large number of IT terms. It's important to note that SE only isn't enough and one needs to define a relation between $|g\rangle$ and w^Q - experimental distributions to construct the complete physical theory. Note than in QM TSE oscillations only approximately described by $\cos kx + \alpha$, but it doesn't disprove our arguments [3]. Thus QM supposedly corresponds to FM with $n = 2$ and $g(x)$ corresponds to $\Psi(x)$ Dirac state vector in x representation.

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